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UNSTEADY DEFORMATION OF ELASTIC SOLIDS IN THE SHAPE OF A CIRCULAR CYLINDRICAL SEGMENT†

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The unsteady deformation of cylindrical solids is investigated using the dynamic theory of elasticity. Special cases of the general solution are pointed out. Numerical results are presented which reflect the specific feature of the stressed state of an infinitely long thick-walled cylinder, which is subjected to plane nonaxi-symmetrical loading. The method of investigating unsteady wave processes in cylindrical solids is similar to that described previously [1–3]. © 1998 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM AND RELATIONS OF GENERAL FORM

Consider, in the general case, an isotropic elastic solid in cylindrical coordinates $(r, \theta\ z)$, bounded by cylindrical surfaces $r = R_0$ and $r = R_1$, the planes $z = 0$ and $z = z_0$ and the half-planes $\theta = 0$ and $\theta = \theta_0$ $(R_0 \leqslant r \leqslant R_1;$ $0 \leqslant z \leqslant z_0;\ 0 \leqslant \theta \leqslant \theta_0)$. We will assume the system of initial conditions to be zero.

When there are no mass forces, the Lamé equations, which describe the motion of a uniform isotropic elastic medium, are equivalent, in circular cylindrical coordinates, to the following system of equations [4]

$$\Delta\varphi = \frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2},\quad \Delta\psi_\alpha = \frac{1}{b^2}\frac{\partial^2\psi_\alpha}{\partial t^2},\quad \alpha = 1,2 \tag{1.1}$$

$$\mathbf{u} = \operatorname{grad}\varphi + \operatorname{rot}(\psi_1\mathbf{e}_z) + \operatorname{rot}\operatorname{rot}(\psi_2\mathbf{e}_z) \tag{1.2}$$

where $\mathbf{u}$ is the displacement vector, φ, ψ_1, ψ_2 are the scalar potentials of the displacements, $\mathbf{e}_z$ is the unit vector of the Z axis, and $\mathbf{a}$ and $\mathbf{b}$ are the velocities of propagation of longitudinal and transverse deformation waves in the elastic medium, respectively.

In the problem of pulsed deformation of a cylindrical solid considered, the solution of the wave equations (1.1) are sought in the form of double expansions in axial and angular coordinates (summation is carried out over n and k from zero to infinity)

$$\begin{aligned}
\varphi &= \sum R_{nk}^0(r,t)\, w_n(\theta) \upsilon_k(z)\\
\psi_1 &= \sum R_{nk}^1(r,t)\frac{1}{\mu_n}\frac{dw_n(\theta)}{d\theta}\upsilon_k(z)\\
\psi_2 &= \sum R_{nk}^2(r,t)\, w_n(\theta)\frac{1}{\nu_k}\frac{d\upsilon_k(z)}{dz}\\
\mu_n &= n\pi/\theta_0,\quad \nu_k = k\pi/z_0
\end{aligned} \tag{1.3}$$

Here $w_n(\theta)$, $\upsilon_k(z)$ are known functions of the corresponding coordinates while $R_{nk}^\beta(r, t)$, $(\beta = 0, 1, 2)$ are to be determined. Specific expressions for the functions $w_n(\theta)$, $\upsilon_k(z)$ will be derived later.

Expansions (1.3) are similar to those given in [5].

Substituting expansions (1.3) into (1.2) we obtain formulae for the components of the displacement vector.

On the end surfaces of a cylindrical panel, when the coordinate functions w_n and υ_k are chosen in the form

$$w_n = \cos\mu_n\theta,\quad \upsilon_k = \cos\nu_k z \tag{1.4}$$

the following boundary conditions are satisfied

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$$\sigma_{z\theta}=0,\ \ \sigma_{zr}=0,\ \ u_z=0\ \text{ when }\ z=0,z_0$$
$$\sigma_{\theta z}=0,\ \ \sigma_{\theta r}=0,\ \ u_\theta=0\ \text{ when }\ \theta=0,\theta_0 \tag{1.5}$$

If we choose them in the form $w_n = \sin\mu_n\theta$, $\upsilon_k = \sin\nu_k z$, the following conditions must be satisfied

$$\sigma_z=0,\ \ u_r=0,\ \ u_\theta=0\ \text{ when }\ z=0,z_0$$
$$\sigma_\theta=0,\ \ u_r=0,\ \ u_z=0\ \text{ when }\ \theta=0,\theta_0$$

Henceforth we will confine ourselves to the form of the functions w_n and υ_k as given by (1.4) and, correspondingly, the boundary conditions on the ends (1.5).

The boundary conditions on the cylindrical surfaces can be realized in two forms. When the boundary stresses are specified we have the following expressions

$$\begin{aligned}
&\sigma_r(R_0,\theta,z)=F_1(\theta,z,t),\quad \sigma_r(R_1,\theta,z,t)=F_4(\theta,z,t)\\
&\sigma_{r\theta}(R_0,\theta,z,t)=F_2(\theta,z,t),\quad \sigma_{r\theta}(R_1,\theta,z,t)=F_5(\theta,z,t)\\
&\sigma_{rz}(R_0,\theta,z,t)=F_3(\theta,z,t),\quad \sigma_{rz}(R_1,\theta,z,t)=F_6(\theta,z,t)
\end{aligned} \tag{1.6}$$

where $F_1(\theta, z, t) - F_6(\theta, z, t)$ are known functions.

When the boundary conditions are specified in terms of displacements, similar relations must be satisfied.

When the functions w_n and υ_k are chosen in the form (1.4), expansions (1.3) become double Fourier series in the variables θ and z.

To satisfy boundary conditions (1.6), the functions $F_j(\theta, z, t)$ must also be expanded in similar Fourier series.

2. FUNDAMENTAL RELATIONS IN LAPLACE TRANSFORM SPACE

To proceed further in constructing the solution we need to determine the functions R^{β}_{nk} which occur in (1.3). To do this, we write the wave equations (1.1) in Laplace transform space, denoting the transform by the superscript L. Substituting the representations for φ, ψ_1 and ψ_2 from (1.3) into the equations obtained and taking (1.4) into account, we obtain modified Bessel equations in $R^{\beta L}_{nk}$ ($\beta = 0, 1, 2$). Their general solutions can be written as follows:

$$R^{\beta L}_{nk}=A^{nkL}_{\beta}(S)\,I_{\mu_n}(r(\nu_k^2+S^2/c_\beta^2)^{1/2})+B^{nkL}_{\beta}(S)\,K_{\mu_n}(r(\nu_k^2+S^2/c_\beta^2)^{1/2}) \tag{2.1}$$
$$\beta=0,1,2;\quad c_0=a,\quad c_1=c_2=b$$

Here $A^{nkL}_{\beta}(S)$ and $B^{nkL}_{\beta}(S)$ are arbitrary functions of the transformation parameter S, $I_\lambda(x)$ is the modified Bessel function of imaginary argument with index λ and $K_\lambda(x)$ is the MacDonald function.

3. THE FUNDAMENTAL RELATIONS IN INVERSE-TRANSFORM SPACE

We will now transfer to inverse-transform space. To do this we use the following formulae [6]

$$\begin{aligned}
&I_\mu((S^2+\gamma^2)^{1/2})(2\pi)^{1/2}(S^2+\gamma^2)^{-\mu/2}e^{-S}=\\
&=L\begin{cases}\gamma^{1/2-\mu}(2t-t^2)^{\mu/2-1/4}I_{\mu-1/2}(\gamma(2t-t^2)^{1/2}), & 0<t<2\\ 0, & t>2\end{cases}
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
&\frac{e^{\varkappa S}K_\mu(\varkappa(S^2+\alpha^2)^{1/2})}{(S^2+\alpha^2)^{\mu/2}}=\\
&=L\left[\left(\frac{\pi}{2}\right)^{1/2}\alpha^{1/2-\mu}(t^2+2\varkappa t)^{\mu/2-1/4}\varkappa^{-\mu}J_{\mu-1/2}(\alpha(t^2+2\varkappa t)^{1/2})\right]
\end{aligned} \tag{3.2}$$
$$\operatorname{Re}\mu>1/2,\quad |\arg\varkappa|<\pi$$

where $J_\rho(x)$ is the Bessel function of the first kind with index ρ.

Using formulae (3.1) and (3.2), and also certain standard rules of the operational calculus, we obtain expressions for the functions R^{β}_{nk} ($\beta = 0, 1, 2$) in inverse-transform space

$$R_{nk}^{\beta} = H\left(t - \frac{R_1 - r}{c_\beta}\right) \int_0^{t-(R_1-r)/c_\beta} A_\beta^{nk}(\tau)\, h_{\mu_n}^{\beta 1}\left(r, t - \frac{R_1 - r}{c_\beta} - \tau\right) d\tau + $$
$$+ H\left(t - \frac{r - R_0}{c_\beta}\right) \int_0^{t-(r-R_0)/c_\beta} B_\beta^{nk}(\tau)\, g_{\mu_n}^{\beta 1}\left(r, t - \frac{r - R_0}{c_\beta} - \tau\right) d\tau \tag{3.3}$$

where we have used the following notation

$$h_{\mu_n}^{\beta 1}(r,t) = \int_0^t h_{\mu_n}^{\beta}(r,\tau)\, d\tau, \quad g_{\mu_n}^{\beta 1}(r,t) = \int_0^t g_{\mu_n}^{\beta}(r,\tau)\, d\tau$$

$H(t)$ is the Heaviside function, and the quantities c_β are described in (2.1), where

$$h_{\mu_n}^{\beta}(r,t) = 0, \quad t > \frac{2r}{c_\beta}$$

$$h_{\mu_n}^{\beta}(r,t) = \nu_k^{1/2-\mu_n} r^{1-\mu_n} (tc_\beta)^{\mu_n/2-1/4} \times$$
$$\times (2r - tc_\beta)^{\mu_n/2-1/4} I_{\mu_n-1/2}(\nu_k (tc_\beta)^{1/2}(2r - tc_\beta)^{1/2}), \quad 0 < t < \frac{2r}{c_\beta}$$

$$g_{\mu_n}^{\beta}(r,t) = \left(\frac{\pi}{2}\right)^{1/2} \nu_k^{1/2-\mu_n} c_\beta^{3/4-\mu_n/2} \times$$
$$\times r^{-\mu_n} (t^2 c_\beta + 2rt)^{\mu_n/2-1/4} J_{\mu_n-1/2}(\nu_k (t^2 c_\beta^2 + 2rtc_\beta)^{1/2})$$

4. DERIVATION OF THE SYSTEM OF INTEGRAL EQUATIONS

We substitute (3.3) into the expressions obtained for the components of the displacement vector, and we substitute the latter into the boundary conditions (1.6). Relations (1.6), after separating the coordinates θ and z, are then converted into a system of six Volterra integral equations in time for the unknown functions $A_\beta^{nk}(t)$, $B_\beta^{nk}(t)$ ($\beta = 0, 1, 2$). We will use a numerical approach to solve these equations, the basic principle of which consists of substituting approximating expressions for the required functions into them, of the following form [3]

$$A_j^{nk}(t) = \sum_{p=1}^{m} A_{jp}^{nk} \Delta_p H, \quad B_j^{nk}(t) = \sum_{p=1}^{m} B_{jp}^{nk} \Delta_p H,$$
$$\Delta_p H = H(t - t_{p-1}) - H(t - t_p) \tag{4.1}$$
$$A_{jp}^{nk} = \text{const}, \quad B_{jp}^{nk} = \text{const}, \quad t_p = p\,\Delta t$$

where Δt is the time step, and $t < m\,\Delta t$, $m = 1, 2, \ldots$.

Substituting (4.1) into the above integral equations, we obtain a system of algebraic equations, recurrent with respect to the index m, for determining the quantities A_{jm}^{nk}, B_{jm}^{nk} ($j = 1, 2, 3, m = 1, 2, \ldots$) which approximate the required functions of time.

Note that the above reduction to a system of algebraic equations using approximations of the required functions when analysing Volterra integral equations is one of a variety of methods for the numerical solution of these equations. Approximation (4.1) naturally defines the sudden change in the stresses which develop in an elastic solid when pulsed loads act on it, and ensures stability of the numerical solution of the integral equations with continuous or integrable kernels.

Converting the formulae obtained for the coefficients of expansion of the displacements and stresses, using approximations (4.1), we obtain relations convenient for numerical realization.

5. DIFFERENT VERSIONS OF THE DEFORMATION OF CYLINDRICAL SOLIDS

Version 1. Using the above formulae we can calculate the effect of an arbitrary pulse on a thick-walled cylindrical panel.

Moreover, special cases, defined by specific values of n and k in (1.3) for the displacement potentials, are of particular interest.

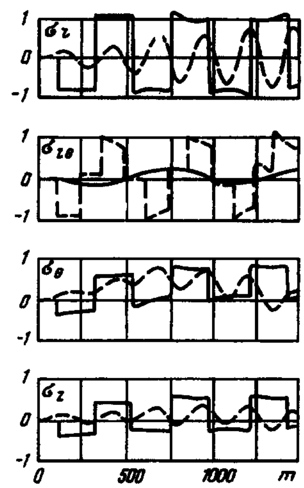

Fig. 1.

Version 2. Suppose all the coefficients in series (1.3) are equal to zero, apart from those corresponding to $n = 0$ and $k = 0$. As follows from the formulae for the expansions of the displacement vector in double Fourier series, this case corresponds to a purely radial shift $u^{00} \equiv u = \partial R^0/\partial r$, which depends only on the radial coordinate and time. This case is equivalent to plane axisymmetric deformation of a closed circular cylindrical layer.

Version 3. Assume that only coefficients with subscripts $k = 0$ and $n \neq 0$ in (1.3) are non-zero. Then $u_z \equiv 0$, while the remaining components of the displacement vector are independent of the z coordinate. This case is equivalent to plane non-axisymmetric deformation of a closed circular cylindrical layer.

Version 4. The case when coefficients with subscripts $n = 0$ and $k \neq 0$ are non-zero corresponds to $u_\theta \equiv 0$, while the remaining components are independent of the variable θ. This version corresponds to axisymmetric deformation of a closed circular cylindrical layer of finite length.

6. NUMERICAL RESULTS

We will give numerical results for the plane non-axisymmetric deformation of a thick-walled cylinder, corresponding to the case when the boundary stresses are specified on the boundary surfaces (Version 3). We will assume that in series (1.3) coefficients with subscripts $n = 2$ and $k = 0$ are non-zero. Figure 1 shows the time-dependence of the dimensionless stresses σ_r^{20}, $\sigma_{r\theta}^{20}$, σ_θ^{20}, σ_z^{20}, calculated at a point in the middle of a thick-walled cylinder, having the following parameters: $R_0 = 0.09$ m, $R_1 = 0.105$ m, $E = 2.058 \times 10^{11}$ N/m^2, $\nu = 0.3$ and $\rho = 7.8 \times 10^3$ kg/m^3, where E, ν and ρ are constants of the material.

It is assumed that the external boundary surface is load-free, while the internal surface is subjected to a finite impulse or radial load: $\sigma_r^{20}(R_0, t) = -\sigma_0 H(\omega_0 - t)$, $\sigma_{r\theta}^{20}(R_0, t) = 0$, where $\omega_0 = (R_1 - R_0)/a$ (the corresponding results are represented by the continuous curves), or a tangential load $\sigma^{20}(R_0, t) = 0$, $\sigma_{r\theta}^{20}(R_0, t)$ (the dashed curves). Along the horizontal time axis we have plotted the number of time steps m: in the first case $\Delta t = 1.13 \times 10^{-8}$ s while in the second case $\Delta t = 1.8 \times 10^{-8}$ s.

The jumps in the values of the stresses due to the sudden nature of the behaviour of the load and also the superposition of deformation waves, reflected from the boundary surfaces of the cylinder, are well tracked. One can also see the specific features of the deformation when radial or tangential boundary stresses are specified.

In conclusion we note that it is possible to extend this method to the solution of problems of pulsed deformation of multilayered cylindrical solids.

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